Percolation Games:
A bridge between Game Theory and Analysis

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Messages

1. PDEs may have a game interpretation
2. The game interpretation is useful in Analysis
3. Some games in the grid have a limit value

PDEs describe Games

Infinity Laplacian

Let an open domain $D \subset \mathbb{R}^d$. Consider $g\colon \partial D \to \mathbb{R}$ a continuous function.

Problem

Compute $u\colon \overline{D} \to \mathbb{R}$ such that

- *$u = g$ on ∂D.*
- *u has minimal $||\nabla u||_\infty$*

Example in one dimension

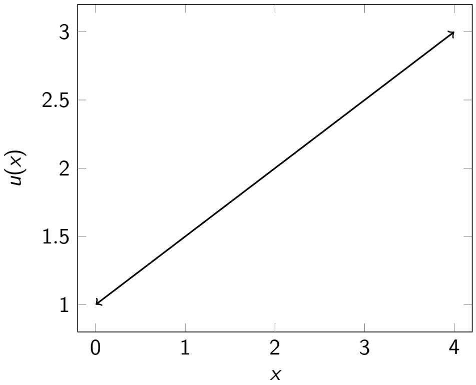

Figure 1: Infinity laplacian in one dimension

Equation description

Let an open domain $D \subset \mathbb{R}^d$. Consider $g \colon \partial D \to \mathbb{R}$ a continuous function.

Problem

Solve the following equation

$$\begin{cases} \Delta_\infty u(x) = \sum_{i,j} \partial^2_{i,j} u(x)\ \partial_i u(x)\ \partial_j u(x) \ = 0 & x \in D \\ u(x) = g(x) & x \in \partial D \end{cases}$$

Game description: Continuous Weigthed Reachability

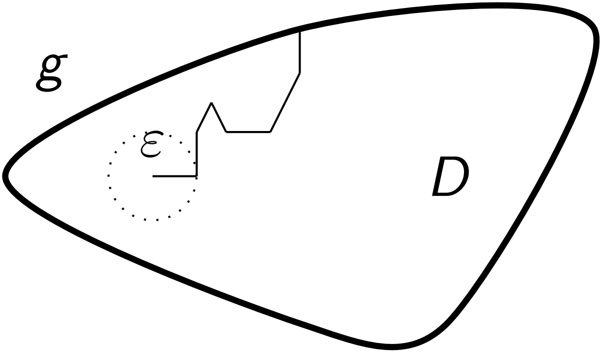

Figure 2: Continuous space reachability games

Dynamic

Let $\varepsilon > 0$. Consider the following dynamic

- State space: $\overline{D}$
- Reward function $g\colon \partial D \to \mathbb{R}$
- Initial position $x \in D$
- Infinite random turn-based game
- At each turn, the corresponding player chooses where to move the state within $B(x, \varepsilon) \cap \overline{D}$
- When arriving at $x \in \partial D$, min-player pays $g(x)$ to the max-player

Dynamic programming property

Let $u^{(\varepsilon)} \colon \overline{D} \to \mathbb{R}$ be the value. Then, for $x \in D$,

$$u^{(\varepsilon)}(x) = \frac{1}{2} \left(\sup_{y \in B(x,\varepsilon) \cap \overline{D}} u^{(\varepsilon)}(y) + \inf_{y \in B(x,\varepsilon) \cap \overline{D}} u^{(\varepsilon)}(y) \right) .$$

Game descrition

Let an open domain $D \subset \mathbb{R}^d$. Consider $g \colon \partial D \to \mathbb{R}$ a continuous function.

Problem

Compute the limit value

$$u(x) := \lim_{\varepsilon \to 0} u^{(\varepsilon)}(x) \,.$$

Infinity Laplacian

From Wikipedia, the free encyclopedia

In mathematics, the **infinity Laplace** (or L^∞-Laplace) operator is a 2nd-order partial differential operator, commonly abbreviated Δ_∞. It is alternately defined by

$$\Delta_\infty u(x) = \langle Du, D^2u\, Du\rangle = \sum_{i,j} \frac{\partial^2 u}{\partial x_i\, \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

or

$$\Delta_\infty u(x) = \frac{\langle Du, D^2u\, Du\rangle}{|Du|^2} = \frac{1}{|Du|^2} \sum_{i,j} \frac{\partial^2 u}{\partial x_i\, \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

The first version avoids the singularity which occurs when the gradient vanishes, while the second version is homogeneous of order zero in the gradient. Verbally, the second version is the **second derivative in the direction of the gradient**. In the case of the infinity Laplace equation $\Delta_\infty u = 0$, the two definitions are equivalent.

While the equation involves second derivatives, usually (generalized) solutions are not twice differentiable, as evidenced by the well-known Aronsson solution $u(x,y) = |x|^{4/3} - |y|^{4/3}$. For this reason the correct notion of solutions is that given by the viscosity solutions.

Viscosity solutions to the equation $\Delta_\infty u = 0$ are also known as **infinity harmonic functions**. This terminology arises from the fact that the infinity Laplace operator first arose in the study of absolute minimizers for $\|Du\|_{L^\infty}$, and it can be viewed in a certain sense as the limit of the p-Laplacian as $p \to \infty$. More recently, viscosity solutions to the infinity Laplace equation have been identified with the payoff functions from **randomized tug-of-war** games. The game theory point of view has significantly improved the understanding of the partial differential equation itself.

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of the partial differential equation itself.

Question

What PDEs have a game interpretation?

Differential games

Let $T > 0, x_0 \in \mathbb{R}^d$.

- Dynamic

$$\begin{cases} \dot{x}(t) = f(x(t), a(t), b(t)) \\ x(0) = x_0 \end{cases}$$

- Payoff

$$\int_0^T g(x(s), a(s), b(s)) ds + g_0(x_T)$$

- Value

$u(T, x_0)$ value of the game.

Hamilton-Jacobi equations

The value function u satisfies

$$\begin{cases} \partial_t u(t,x) + H(\nabla u(t,x), x) = 0 \\ u(0,x) = g_0(x) \end{cases}$$

where

$$H(p,x) := -\sup_{a \in A} \inf_{b \in B} \{ g(x,a,b) + f(x,a,b) \cdot p \} .$$

Example: Collecting coins

The *environment* might be as follows.

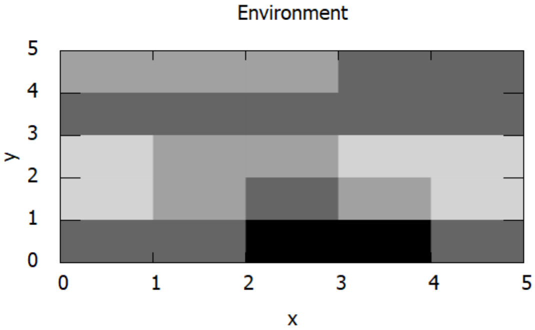

Figure 3: Environment for collecting coins

Example: Collecting coins

- Dynamic

$$\begin{cases} \dot{x}_1(t) = b(t) \\ \dot{x}_2(t) = a(t) \\ x(0) = x_0 = (0,0) \end{cases}$$

- Payoff

$$\int_0^1 g(x(s))ds$$

- Value $u(1,0)$ is the aggregation of coins the max-player can get in one unit of time.

Games assists Analysis

Homogenization

Let $H \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a *"hamiltonian"*. Define, for $\varepsilon > 0$,

$$\begin{cases} \partial_t u^{(\varepsilon)} + H(\nabla u^{(\varepsilon)}, x/\varepsilon) = 0 & x \in D \\ u^{(\varepsilon)}(0,x) = u_0(x) & x \in D \end{cases}$$

Definition (Homogenization)

The hamiltonian H *homogenizes* if

1. $(u^{(\varepsilon)}) \xrightarrow[\varepsilon \to 0]{} u$
2. u is the solution of

$$\begin{cases} \partial_t u + \overline{H}(\nabla u) = 0 & x \in D \\ u(0,x) = u_0(x) & x \in D \end{cases}$$

where $\overline{H}$ is the *effective* hamiltonian.

Example: Collecting coins

The corresponding Hamiltonian is

$$\begin{aligned} H(p,x) &= -\sup_{a\in A}\inf_{b\in B}\{g(x,a,b)+f(x,a,b)\cdot p\} \\ &= -\sup_{a\in[-1,1]}\inf_{b\in[-1,1]}\{g(x)+bp_1+ap_2\} \\ &= -(g(x)-|p_1|+|p_2|)\,. \end{aligned}$$

Homogenization question

Theorem (Sufficient conditions)

If H is continuous and

- *periodic in x, i.e.* $H(p, x + x_0) = H(p, x)$
- *coercive in p (uniformly in x), i.e.* $\lim_{||p||_\infty \to \infty} H(p, x) = \infty$

then H homogenizes.

Stochastic Homogenization

Let $H\colon \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a *random* hamiltonian. Define, for $\varepsilon > 0$,

$$(PDE_\varepsilon) \begin{cases} \partial_t u^{(\varepsilon)} + H(\nabla u^{(\varepsilon)}, x/\varepsilon, \omega) = 0 & x \in \Omega \\ u^{(\varepsilon)}(0, x) = u_0(x) & x \in \Omega \end{cases}$$

Definition (Stochastic Homogenization)

The random Hamiltonian H *homogenizes* if

1. $(U^{(\varepsilon)}) \xrightarrow[\varepsilon \to 0]{} u$
2. u is the solution of

$$\begin{cases} \partial_t u + \overline{H}(\nabla u) = 0 & x \in \Omega \\ u(0, x) = u_0(x) & x \in \Omega \end{cases}$$

where $\overline{H}$ is the *effective* (deterministic) hamiltonian.

Going back to games

Assume that u_0 is linear and the equation $(PDE_{\varepsilon=1})$ has a (random) solution. Then,

$$U^{(\varepsilon)}(t,x) := \varepsilon U^{(1)}\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

is a solution of (PDE_{ε}).
In particular, if H homogenizes,

$$\begin{aligned} u(1,0) &= \lim_{\varepsilon \to 0} \varepsilon U^{(1)}\left(\frac{1}{\varepsilon}, \frac{0}{\varepsilon}\right) \\ &= \lim_{T \to \infty} \frac{1}{T} U^{(1)}(T, 0) \,. \end{aligned}$$

Going back to games

If the equation (PDE_ε) has a game interpretation, we would have

Lemma

If H homogenizes, then the corresponding (random) differential game has a deterministic limit value.

Question

What games in $\mathbb{R}^d$ (or the grid $\mathbb{Z}^d$) have a limit value?

Random game on the plane

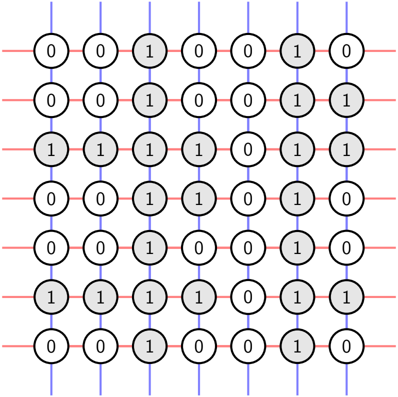

Figure 4: Average payoff game in random media

Dynamic

Consider the following dynamic

- State space is $\mathbb{Z}^2$
- Random reward function $G \colon \mathbb{Z}^2 \to \mathbb{R}$, where $G(z) \sim B(p)$, for $p \in [0, 1]$
- Initial state is $(0, 0)$
- Infinite turn-based game
- At each turn, the corresponding player chooses where to move the state:
  - Max-player chooses *up* or *down*
  - Min-player chooses *left* or *right*
- The reward is the average reward after n stages

$$\frac{1}{n} \sum_{m=1}^{n} G(z_m) \,.$$

Percolation thresholds

Denote the (random) mean value V_n

Theorem

There exists $0 < p_0 < p_1 < 1$ *such that*

$$
\begin{aligned}
(V_n) &\xrightarrow[n \to \infty]{} 0 && \forall p < p_0 \\
(V_n) &\xrightarrow[n \to \infty]{} 1 && \forall p > p_1
\end{aligned}
$$

Transient variant

Consider the game that restricts the players to (essentially) never enter $B(0, \sqrt{m})$ at stage m.

Theorem

For all $p \in [0, 1]$, there exists a limit value v_∞, i.e.

$$(V_n) \xrightarrow[n \to \infty]{} v_\infty \,.$$

(for now, we show convergence in probability)

Transient variant, details

Let $\varepsilon > 0$. Define

$$Z_m \approx \{z \in \mathbb{Z}^2 : ||z||_2 \leq m^{(1+\varepsilon)1/2} - 1\},$$

where the detail is that $|Z_{m+1} \setminus Z_m| = 1$.
Restrict the players from entering Z_m at stage m. Then,

$$(V_n) \text{ is very close to } v_\infty .$$

Proof steps

- V_n concentrates on its expectation $\mathbb{E}(V_n)$
- $(\mathbb{E}(V_n)) \xrightarrow[n\to\infty]{} v_\infty$
- $(\mathbb{E}(V_n))_{n\in\mathbb{N}}$ converge fast to v_∞
- Therefore, V_n concentrates on v_∞

The proof technique does not generalize if there is a lot of dependence from the past.

Why transient?

We rely on Azuma's inequality.

Lemma (Concentration of martingales)

Let $(X_n)_{n\in\mathbb{N}}$ be a martingale and $(c_n)_{n\in\mathbb{N}}$ a real sequence such that, for all $n \in \mathbb{N}$, $|X_n - X_{n+1}| \leq c_n$ almost surely. Then, for all $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\mathbb{P}(|X_n - X_0| \geq \varepsilon) \leq 2\exp\left(\frac{-\varepsilon^2}{2\sum_{m=0}^{n-1} c_m^2}\right) .$$

Proof: Concentration on $\mathbb{E}(V_n)$

We aim to show that $\mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon)$ decreases with n. We will do so by defining a martingale and applying Azuma's inequality.

Proof: Concentration on $\mathbb{E}(V_n)$ (2)

For $m \in \mathbb{N}$,

- Define the σ-algebra

$$\mathcal{C}_m := \sigma(\{G(z,i,j) : z \in Z_m, i \in I, j \in J\}) \, .$$

- Note the inequality

$$|\mathbb{E}(V_n(0)|\mathcal{C}_m) - \mathbb{E}(V_n(0)|\mathcal{C}_{m+1})| \leq \begin{cases} \frac{1}{n} & m < n^{2/(1+\varepsilon)} \\ 0 & m \geq n^{2/(1+\varepsilon)} \end{cases} \, .$$

- Define the martingale

$$X_m := \mathbb{E}(V_n(0)|\mathcal{C}_m) \, .$$

Proof: Concentration on $\mathbb{E}(V_n)$ (3)

Denote $m(n) = n^{2/(1+\varepsilon)}$. Then, applying Azuma's inequality,

$$\begin{aligned}
\mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon) &= \mathbb{P}(|X_{m(n)} - X_0| \geq \varepsilon) \\
&\leq 2 \exp\left(\frac{-\varepsilon^2}{2 \sum_{m=0}^{n-1} (1/n)^2} \right) \\
&\leq 2 \exp\left(\frac{-\varepsilon^2 n^2}{2m(n)} \right) \\
&\leq 2 \exp\left(\frac{-\varepsilon^2}{2} n^{2\varepsilon/(1+\varepsilon)} \right) .
\end{aligned}$$

Therefore, V_n concentrates on $\mathbb{E}(V_n)$.

Proof steps

- V_n concentrates on its expectation $\mathbb{E}(V_n)$
- $(\mathbb{E}(V_n)) \xrightarrow[n\to\infty]{} v_\infty$
- $(\mathbb{E}(V_n))_{n\in\mathbb{N}}$ converge fast to v_∞
- Therefore, V_n concentrates on v_∞

Proof: Convergence of $\mathbb{E}(V_n)$

We aim to show that $\mathbb{E}(V_n)$ converges.
We will use that subadditive sequences converge.

Lemma (Convergence of subadditive sequences)

Let $\phi\colon \mathbb{N} \to (0,\infty)$ *be an increasing function such that* $\sum_{n=1}^{\infty} \phi(n)/n^2 < \infty$, *and* $(f(n))_{n\in\mathbb{N}}$ *be a sequence such that, for all* $n \in \mathbb{N}$ *and all* $m \in [n/2, 2n]$,

$$f(n+m) \leq f(n) + f(m) + \phi(n+m)\,.$$

Then, there exists $L \in \mathbb{R}$ *such that*

$$\left(\frac{f(n)}{n}\right) \xrightarrow[n\to\infty]{} L\,.$$

Proof: Convergence of $\mathbb{E}(V_n)$ (2)

$$
\begin{aligned}
&\mathbb{P}(\exists z \in B_\infty(0,2n) \quad |V_n(z) - \mathbb{E}(V_n)| \geq \varepsilon) \\
&\quad \leq \sum_{z \in B_\infty(0,2n)} \mathbb{P}(|V_n(z) - \mathbb{E}(V_n)| \geq \varepsilon) && \text{(union sum)} \\
&\quad = \sum_{z \in B_\infty(0,2n)} \mathbb{P}(|V_n(0) - \mathbb{E}(V_n)| \geq \varepsilon) && \text{(space-homogeneity)} \\
&\quad \leq |B_\infty(0,2n)| 2 \exp\left(\frac{-\varepsilon^2}{2} n^{2\varepsilon/(1+\varepsilon)} \right) && \text{(Azuma's inequality)} \\
&\quad \leq (4n+1)^d 2 \exp\left(\frac{-\varepsilon^2}{2} n^{2\varepsilon/(1+\varepsilon)} \right) && \text{(Azuma's inequality)} \\
&\quad =: \psi(n, \varepsilon) .
\end{aligned}
$$

Proof: Convergence of $\mathbb{E}(V_n)$ (3)

$$
\begin{aligned}
&\mathbb{E}\left(\min_{z \in B_\infty(0,2n)} V_n(z)\right) \\
&\geq 0\,\mathbb{P}\left(\min_{z \in B_\infty(0,2n)} V_n(z) \leq \mathbb{E}(V_n) - \varepsilon_n\right) \\
&\qquad + (\mathbb{E}(V_n) - \varepsilon_n)\mathbb{P}\left(\min_{z \in Z^{(2n)}} V_n(z) \geq \mathbb{E}(V_n) - \varepsilon_n\right) \\
&\geq (1 - \psi(n, \varepsilon_n))\mathbb{E}(V_n) - \varepsilon_n \\
&\geq \mathbb{E}(V_n) - (\psi(n, \varepsilon_n) + \varepsilon_n)\,.
\end{aligned}
$$

Now we can show that $n\mathbb{E}(V_n)$ is subadditive.

Proof: Convergence of $\mathbb{E}(V_n)$ (4)

By playing by blocks, we obtain, for $m \leq 2n$,

$$
\begin{aligned}
(m+n)\mathbb{E}(V_{m+n}) &\geq m\mathbb{E}(V_m) + n\mathbb{E}(\min_{z \in Z^{(2n)}} V_n(z)) \\
&\geq m\mathbb{E}(V_m) + n\mathbb{E}(V_n) - n(\psi(n, \varepsilon_n) + \varepsilon_n)\,.
\end{aligned}
$$

which is sufficient subadditivity and therefore, there exists v_∞ such that

$$
\mathbb{E}(V_n) \xrightarrow[n \to \infty]{} v_\infty\,.
$$

Proof steps

- V_n concentrates on its expectation $\mathbb{E}(V_n)$
- $(\mathbb{E}(V_n)) \xrightarrow[n\to\infty]{} v_\infty$
- $(\mathbb{E}(V_n))_{n\in\mathbb{N}}$ converge fast to v_∞
- Therefore, V_n concentrates on v_∞

Proof: Fast convergence of $\mathbb{E}(V_n)$

Recall that

$$\mathbb{E}(V_{2n}) \geq \mathbb{E}(V_n) - (\psi(n, \varepsilon_n) + \varepsilon_n) \,.$$

Moreover, we can choose $\delta > 0$ such that

$$(\psi(n, \varepsilon_n) + \varepsilon_n) \in O(n^{-\delta}) \,.$$

Proof: Fast convergence of $\mathbb{E}(V_n)$

By the telescopic sum, we get for $\ell > 0$

$$\begin{aligned}
\mathbb{E}(V_{2^\ell n}) &\geq \mathbb{E}(V_n) - \sum_{\ell'=0}^{\ell-1} \mathbb{E}\left(V_{2^{\ell'} n}\right) - \mathbb{E}\left(V_{2^{\ell'+1} n}\right) \\
&\geq \mathbb{E}(V_n) - \sum_{\ell'=0}^{\ell-1} K(2^{\ell'} n)^{-\delta} \\
&\geq \mathbb{E}(V_n) - n^{-\delta} \frac{K}{1 - 2^{-\delta}} \geq \mathbb{E}(V_n) + O(n^{-\delta})\,.
\end{aligned}$$

Therefore,

$$|v_\infty - \mathbb{E}(V_n)| \in O(n^{-\delta})\,.$$

Proof steps

- V_n concentrates on its expectation $\mathbb{E}(V_n)$
- $(\mathbb{E}(V_n)) \xrightarrow[n\to\infty]{} v_\infty$
- $(\mathbb{E}(V_n))_{n\in\mathbb{N}}$ converge fast to v_∞
- Therefore, V_n concentrates on v_∞

Proof: Concentration on v_∞

Recall that

- $|v_\infty - \mathbb{E}(V_n)| \in O(n^{-\delta})$
- $\mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon) \leq \exp\left(\frac{-\varepsilon^2}{2} n^{2\varepsilon/(1+\varepsilon)}\right)$

Therefore, there exists $K > 0$ such that

$$\begin{aligned}
&\mathbb{P}(|V_n - v_\infty| \geq \varepsilon + Kn^{-\delta}) \\
&\quad \leq \mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon + Kn^{-\delta} - |\mathbb{E}(V_n) - v_\infty|) \\
&\quad \leq \mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon) \\
&\quad \leq \exp\left(\frac{-\varepsilon^2}{2} n^{2\varepsilon/(1+\varepsilon)}\right) .
\end{aligned}$$

Transient variant, formalization

Let $\varepsilon > 0$. Define

$$Z_m \approx \{z \in \mathbb{Z}^2 : ||z||_2 \leq m^{(1+\varepsilon)1/2} - 1\} .$$

Restrict the players from entering Z_m at stage m. Then, there exists $K, \delta > 0$ such that for all $\varepsilon > 0$

$$\mathbb{P}(|V_n - v_\infty| \geq \varepsilon + Kn^{-\delta}) \xrightarrow[n \to \infty]{} 0 .$$

Random game on the plane

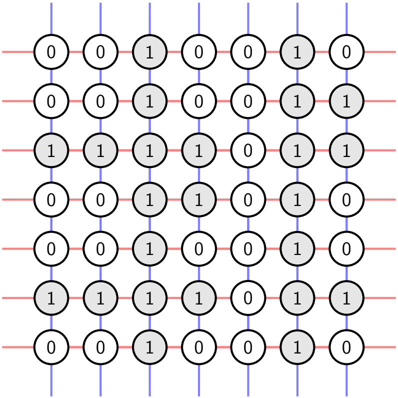

Figure 5: Average payoff game in random media

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